

Discrete-symmetry vortices as angular Bloch modes

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The most general form for symmetric modes of nonlinear discrete-symmetry systems with non-linearity depending on the modulus of the field is presented. Vortex solutions are demonstrated to behave as Bloch modes characterized by an angular Bloch momentum associated to a periodic variable, periodicity being fixed by the order of discrete point-symmetry of the system. The concept of angular Bloch momentum is thus introduced to generalize the usual definition of angular momentum to cases where $O(2)$ -symmetry no longer holds. The conservation of angular Bloch momentum during propagation is demonstrated.

Rotational symmetry in the $x-y$ plane, defined by the $O(2)$ symmetry group, implies the conservation of the z -component of angular momentum in two-dimensional nonlinear systems. If the $O(2)$ continuous symmetry is substituted by a discrete rotational symmetry, the usual approach based on Noether's theorem fails since infinitesimal symmetry transformations are no longer allowed. Conservation of angular momentum is expected to break-down in its usual form and, consequently, nonlinear dynamics should show special features as compared to the $O(2)$ -symmetric case. Since angular momentum is expected to realize differently in discrete-symmetry systems, it is particularly interesting to compare the different behavior of $O(2)$ -symmetric and discrete-symmetry vortices. A general feature of vortex solutions is their characteristic phase dislocation, which is determined by an integer number that will be referred to as vorticity (also known as winding-number, "topological charge" or spin). In $O(2)$ -symmetric systems, rotationally invariant vortices (i.e., vortex solutions whose amplitude is $O(2)$ -symmetric) are eigenfunctions of the angular momentum operator with eigenvalue (angular momentum) given by vorticity. Oppositely, discrete-symmetry vortices have no well-defined angular momentum. Examples of both types of vortices can be found in optics as well as in Bose-Einstein condensate (BEC) systems. Optical vortices with $O(2)$ -symmetry have been experimentally observed in homogeneous self-defocussing nonlinear Kerr media [1] whereas discrete-symmetry optical vortices have been observed in optically-induced photonic lattices [2, 3]. The influence of discrete symmetry in the features of optical vortices are strongly reflected in their angular properties. The phase of a discrete-symmetry vortex presents, besides the typical linear angular dependence characteristic of $O(2)$ -symmetric vortices, an additional sinusoidal contribution completely fixed by the order of the discrete-symmetry of the system (the so-called group phase)[4]. Besides, these systems exhibit a vorticity cutoff equally determined by the order of the discrete-symmetry, in such a way that no vortices of arbitrary order are permitted [5].

The aim of this paper is two-fold. On the one hand,

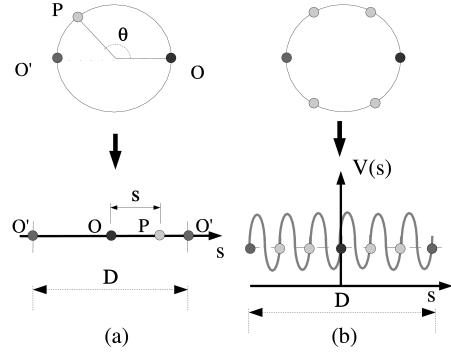


FIG. 1: (a) Mapping of the angular variable θ into the non-compact coordinate s . (b) Equivalent periodic potential using this mapping (for an operator of the type $\tilde{L}(s) = \nabla^2 - V(s)$) corresponding to a system with point-symmetry of order $n = 6$.

I will propose a new approach to the search for vortex solutions of a general nonlinear equation that describes not only the conventional optical and BEC cases but also more general situations. I will prove a general expression for symmetric vortex solutions by showing that they are angular Bloch modes. On the second hand, I will extend the conservation of angular momentum to the discrete-symmetry case by introducing the concept of angular Bloch momentum and by demonstrating its conservation during propagation.

Let us consider the 2D nonlinear eigenvalue equation for a system possessing discrete point-symmetry of the C_n type (discrete rotations of order n):

$$[L_0 + L_{NL}(|\phi|)] \phi = -\mathcal{E}\phi, \quad (1)$$

where L_0 stands for the field-independent part of the full differential operator (linear part) and $L_{NL}(|\phi|)$ for the nonlinear part. There is no restriction about the specific form of L_0 and L_{NL} . Usually, L_0 depends on the transverse position coordinates through gradient operators and explicit functions of position (a typical example is $L_0 = \nabla^2 - V_0(x, y)$). The only assumption made is that the nonlinear part L_{NL} depends on the field

through its modulus exclusively. There can be also an explicit dependence on coordinates through other functions (e.g., inhomogeneous nonlinear coefficients: $L_{\text{NL}} = \gamma_2(x, y)|\phi|^2 + \gamma_4(x, y)|\phi|^4$). It is considered that the system is invariant under discrete rotations \mathcal{C}_n , so that all operators and coefficients defining L_0 and L_{NL} are assumed to be invariant under discrete rotations of n th order. If one is interested in symmetric solutions, i.e., those functions verifying that their modulus is a \mathcal{C}_n -invariant ($|\phi(r, \theta + 2\pi/n)| = |\phi(r, \theta)|$), then the entire operator $L = L_0 + L_{\text{NL}}$ becomes invariant under discrete rotations: $L(r, \theta) = L(r, \theta + \epsilon)$, $\epsilon \equiv 2\pi/n$. One can transform this angular dependence into a dependence on a length variable—defined within an interval of the real axis—by using the mapping $\theta \rightarrow s = \theta D/2\pi$. This mapping maps the unit circle \mathcal{S}_1 (where θ is defined: $\theta \in [-\pi, \pi]$) onto the real axis interval $[-D/2, D/2]$ (see Fig.1). In terms of the new length variable s , invariance under discrete rotations of the L -operator becomes a periodicity property: if $\tilde{L}(r, s) \equiv L(r, 2\pi s/D)$ then $\tilde{L}(r, s + a) = \tilde{L}(r, s)$ where a is the period in the non-compact variable s . According to the previous mapping, the period a is fixed by the discrete-rotation angle ϵ and thus it depends on the order of the discrete symmetry: $a = \epsilon D/2\pi = D/n$. In this way, Eq.(1) for symmetric functions is transformed into $\tilde{L}(r, s)\varphi(r, s) = -\mathcal{E}\varphi(r, s)$ where $\varphi(r, s) \equiv \phi(r, 2\pi s/D)$. One has thus converted the original discrete-symmetry eigenvalue problem into a problem of finding the self-consistent modes of a periodic operator. The modulus of these modes are invariant under periodic translations ($|\varphi(r, s + a)| = |\varphi(r, s)|$), a consequence of the invariance of the modulus of a symmetric solution under discrete-rotations. As a clarifying example, one can consider a standard operator of the type $\tilde{L}(r, s) = \nabla^2 - V(r, s)$, where $V(r, s) = V_0(r, s) + V_{\text{NL}}(|\varphi(r, s)|)$. If $\varphi(r, s)$ is a solution of $\tilde{L}\varphi = -\mathcal{E}\varphi$ with periodic modulus then it has to be an eigenmode of the periodic operator generated by itself $\tilde{L}(r, s) = \nabla^2 - V(r, s)$. According to Bloch's theorem, since V is periodic $V(r, s + a) = V(r, s)$ the solution of an eigenvalue equation of the previous type is given by Bloch modes in the periodic variable s . This argument equally applies to the general periodic operator $\tilde{L}(r, s)$, so that the most general solution of $\tilde{L}\varphi = -\mathcal{E}\varphi$ is given by 1D Bloch modes [6]: $\varphi_{p\nu}(r, s) = e^{ips}u_{p\nu}(r, s)$, where p is the Bloch momentum or pseudo-momentum of the mode and u_p is the so-called Bloch function. The index ν is known as the band index and occurs because, at a given value of p , there are many different eigenmodes of L . Since ν does not play a role in the current discussion, I will omit it from the notation (although one has to recall that it is always present). The Bloch function is a periodic function of s with the periodicity of the \tilde{L} operator: $u_p(r, s + a) = u_p(r, s)$.

One can obtain interesting properties of solutions of Eq.(1) by re-interpreting well-known properties of 1D Bloch modes. A Bloch mode is characterized by the value

of its Bloch momentum p . However, unlike plane waves, p is not an eigenvalue of the momentum operator and its value is constrained to lie in a restricted interval, called the Brillouin zone, defined by the condition $|p| \leq \pi/a$. Moreover, due to the definition of the length coordinate s in terms of the angular variable θ , the φ_p function has to be additionally periodic in the interval length D ($s \in [-D/2, D/2]$), $\varphi_p(r, s) = \varphi_p(r, s + D)$. As a consequence, a discretization condition for the Bloch momentum p is obtained: $e^{ipD} = 1 \Rightarrow p = p_m = 2\pi m/D$, $m \in \mathbb{Z}$. If one reverses the $\theta \rightarrow s$ mapping by re-introducing the angular variable θ in $\varphi_p(r, s) = e^{ips}u_p(r, s)$, one obtains the solution in its original form:

$$\phi_m(r, \theta) = e^{im\theta}\tilde{u}_m(r, \theta), \quad m \in \mathbb{Z}, \quad |m| \leq n/2. \quad (2)$$

I shall call these solutions angular Bloch modes. They constitute the general symmetric solutions of the nonlinear eigenvalue equation (1). Because of the angular nature of the θ variable and its relation to the Bloch momentum $p_m = 2\pi m/D$, the index m will be referred to as the angular Bloch momentum (or pseudo angular-momentum) of the angular Bloch mode. The restriction on the permitted values of the angular Bloch momentum m is a consequence of the Brillouin zone limitation. The existence of the Brillouin zone for the Bloch momentum p establishes a condition for its permitted values: $|p| \leq \pi/a$. Since p is discretized according to $p_m = 2\pi m/D$, $m \in \mathbb{Z}$, the Brillouin zone limitation effectively imposes a strict constraint into the angular Bloch momentum $|m| \leq D/2a$. Inasmuch as the period a is determined by the order of symmetry n ($a = D/n$), one finds that the modulus of the angular Bloch momentum presents the upper bound $|m| \leq n/2$ occurring in Eq.(2).

The expression (2) is the same one that it is obtained for vortex solutions in a nonlinear system with point-symmetry \mathcal{C}_n using group theory arguments [4, 5]: $\phi_{\bar{l}}(r, \theta) = e^{i\bar{l}\theta}\phi_0^{(\bar{l})}(r, \theta)$, $\bar{l} \in \mathbb{Z}$ being the index of the group representation where the solution belongs to and $\phi_0^{(\bar{l})}$ a discrete-rotation invariant function, $\phi_0^{(\bar{l})}(r, \theta + 2\pi/n) = \phi_0^{(\bar{l})}(r, \theta)$. The invariant function $\phi_0^{(\bar{l})}$ predicted by group theory is nothing but the Bloch function \tilde{u}_m of the angular Bloch mode (2). Vorticity equals the index of the representation, so that $\phi_{\bar{l}}$ represents a vortex with vorticity $v = \bar{l}$ (the exception is the $\bar{l} = n/2$ —even n —solution)[5]. The comparison of $\phi_{\bar{l}}(r, \theta) = e^{i\bar{l}\theta}\phi_0^{(\bar{l})}(r, \theta)$ and Eq.(2) yields to interesting equivalences. Vortex solitons of vorticity $v = \bar{l}$ appear now as angular Bloch modes carrying angular Bloch momentum $m = v = \bar{l}$ ($m = n/2$ —even n —excluded). The restriction on angular Bloch momentum $|m| \leq n/2$, along with the previously established relation between vorticity and angular Bloch momentum, fixes a cutoff for the vorticity: $|v| \leq n/2$ ($v = n/2$ —even n —excluded). This condition

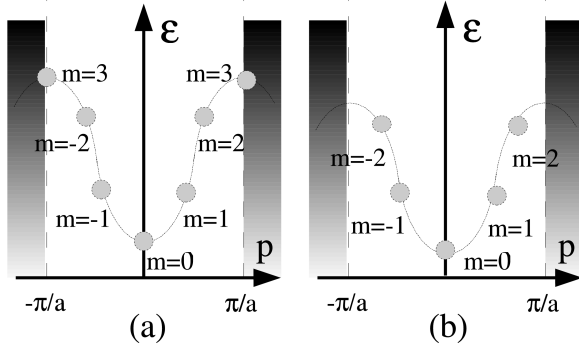


FIG. 2: Schematic representation of Brillouin zones for two different order of symmetries: (a) even n ($n = 6$); (b) odd n ($n = 5$). Permitted angular Bloch modes are shown.

can be also rephrased as:

$$|v| < n/2 \text{ (even } n) \text{ and } |v| \leq (n-1)/2 \text{ (odd } n). \quad (3)$$

The explicit distinction between even and odd orders is made to remark the different behavior existent in the corresponding Brillouin zones in both cases. For odd n , the discretized Bloch momentum p_m can never achieved the zone limit $|p_m| < \pi/a$, whereas this is certainly possible for even n . This can be clearly visualized in Fig. 2, where two examples of Brillouin zones for even and odd n are shown. It is remarkable that the vorticity cut-off expressed in Eq.(3) has been obtained in a completely different mathematical framework using group-theory arguments [5].

At this point, it is interesting to compare the discrete-symmetry situation with that corresponding to continuous rotational symmetry. In the $O(2)$ case, Noether's theorem ensures the conservation of the z -component of the angular momentum associated to the ϕ field during propagation:

$$j_z = \int_{\mathbb{R}^2} dx dy \phi^*(\mathbf{r})(\mathbf{r} \wedge \nabla)_z \phi(\mathbf{r}) \Rightarrow \frac{dj_z(z)}{dz} = 0. \quad (4)$$

This quantity is also the expectation value of the angular momentum operator $L_z = -i\partial/\partial\theta$ for the ϕ field, i.e., $j_z = \langle \phi | L_z | \phi \rangle$. Since $O(2)$ -symmetric vortex solutions are of the type $\phi_{\bar{l}}(r, \theta) = e^{i\bar{l}\theta} f_{\bar{l}}(r)$, vortex solutions are eigenfunctions of the angular momentum operator with eigenvalue \bar{l} . Therefore, j_z equals \bar{l} for (normalized to one) vortex solutions and both represent the same constant of motion. On the other hand, the vorticity v of $O(2)$ -symmetric vortices is directly given by \bar{l} : $v = j_z = \bar{l}$. Thus, angular-momentum and vorticity are equivalent quantities for $O(2)$ -symmetric vortices.

In a discrete-symmetry system the z -component of angular momentum j_z (Eq.(4)) is no longer conserved. Noether's theorem does not apply because the symmetry under consideration ceases from being continuous. It is

then necessary to find a different conserved quantity that explicitly manifests the discrete-symmetry invariance of the system. It will be proved next that this quantity is nothing but the angular Bloch momentum. In order to carry out the previous demonstration, it is simpler to resort to the representation of Bloch modes in terms of the non-compact variable s . It will be assumed that evolution is provided by a first-order operator in the evolution variable z , i.e., $\tilde{L}\varphi(r, s, z) = -i\partial\varphi(r, s, z)/\partial z$. This type of evolution operator includes Schroedinger-like equations (like those appearing in nonlinear paraxial evolution in optical systems or in the evolution of Bose-Einstein condensates) but also more general evolution equations, as non-paraxial forward equations in axially-invariant optical systems [7]. My aim now is to analyze the evolution of a field amplitude whose initial condition is given by a Bloch mode with well-defined pseudo-momentum p , i.e., $\varphi(r, s, 0) = \varphi_p(r, s) = e^{ips} u_p(r, s)$. It is important to stress that this mode does not need to be a stationary solution of the vortex type satisfying the nonlinear eigenvalue Eq.(1), so that its amplitude can evolve with z . It is a known fact that Bloch momentum is conserved during propagation in a periodic linear system. However, this question is not obvious in the nonlinear case inasmuch as the evolution operator explicitly depends on the field itself.

Let us consider the total evolution from 0 to z as a succession of infinitesimal evolution steps of length ϵ in the limit $\epsilon \rightarrow 0$. In a first-order evolution equation, the infinitesimal evolution of a field amplitude from the axial slice z_j to the following slice $z_{j+1} = z_j + \epsilon$ is given by:

$$\varphi_{j+1}(r, s) = e^{i\tilde{L}(|\varphi_j|)\epsilon} \varphi_j(r, s). \quad (5)$$

Now I will prove that if φ_0 ($\varphi_0 \equiv \varphi(r, s, 0)$) is a Bloch mode with well-defined Bloch momentum p , then φ_j is also a Bloch mode with the same pseudomomentum p for all values of j . The proof is carried out by induction. Let us start by calculating the pseudo-momentum of the $j = 1$ amplitude φ_1 . The initial amplitude φ_0 is a Bloch mode with pseudo-momentum p ($\varphi_0 = \varphi_p(r, s) = e^{ips} u_p(r, s)$) and thus it is an eigenfunction of the translation operator T_a : $T_a \varphi_0(r, s) = \varphi_0(r, s + a) = e^{ipa} \varphi_0(r, s)$. The amplitude $|\varphi_0|$ is a periodic function of s with periodicity a since it is the modulus of the periodic Bloch function $u_p(r, s)$: $|\varphi_0(r, s + a)| = |\varphi_0(r, s)|$, so that the nonlinear part of the \tilde{L} operator is translational invariant, $L_{NL}(|\varphi_0(r, s + a)|) = L_{NL}(|\varphi_0(r, s)|)$. Therefore the full operator \tilde{L} is invariant under finite translations since both its linear and nonlinear part are invariant, $[\tilde{L}(|\varphi_0|), T_a] = 0$. This implies in turn that the infinitesimal evolution operator in Eq.(5) commutes with T_a : $[e^{i\tilde{L}(|\varphi_0|)\epsilon}, T_a] \stackrel{\epsilon \rightarrow 0}{=} [1 + i\tilde{L}(|\varphi_0|)\epsilon, T_a] = 0$. Now I apply the T_a operator onto φ_1 and take into account Eq.(5) for $j = 0$ to find $T_a \varphi_1 = e^{i\tilde{L}(|\varphi_0|)\epsilon} T_a \varphi_0 = e^{ipa} \varphi_1$, where I have used the fact that the evolution and translation

operator commute between them and that φ_0 is an eigenfunction of T_a . Consequently, φ_1 is a Bloch mode with pseudo-momentum p . Now, following the induction procedure, I will assume the Bloch-mode property to be true for φ_j and I will prove it for φ_{j+1} . The demonstration is analogous to the $j = 0$ case. I assume φ_j is a Bloch-mode with pseudo-momentum p . Then its modulus is translational invariant, so $\tilde{L}(|\varphi_j|)$ and $e^{i\tilde{L}(|\varphi_j|)\varepsilon}$ are: $[\tilde{L}(|\varphi_j|), T_a] = [e^{i\tilde{L}(|\varphi_j|)\varepsilon}, T_a] = 0$. If one acts with the translation operator T_a onto φ_{j+1} , commutes the translation and evolution operators and takes into account that $T_a\varphi_j = e^{ipa}\varphi_j$ (Bloch-mode condition for φ_j), one readily finds that $T_a\varphi_{j+1} = e^{ipa}\varphi_{j+1}$. This shows that φ_j is a Bloch-mode with pseudo-momentum p for all values of j if the initial amplitude φ_0 is. Expressed in different words: axial nonlinear evolution preserves the Bloch momentum p .

The continuous limit $\varepsilon \rightarrow 0$ of the previous statement permits to give an expression for the evolving field amplitude $\varphi(r, s, z)$: if $\varphi(r, s, 0) = e^{ips}u_p(r, s)$ then it is also true that $\varphi(r, s, z) = e^{ips}u_p(r, s, z)$, u_p being a z -dependent Bloch function. If we reintroduce the angular variable θ instead of the non-compact one s , one obtains that the general expression for the evolution of an initial angular Bloch mode will well-defined angular Bloch momentum m in the system under consideration is:

$$\phi_m(r, \theta, z) = e^{im\theta}\tilde{u}_m(r, \theta, z), \quad m \in \mathbb{Z}, \quad |m| \leq n/2. \quad (6)$$

This equation establishes that the angular Bloch momentum m is preserved by nonlinear evolution.

The interplay between ordinary angular momentum and angular Bloch momentum can be visualized by calculating the expectation value of the angular momentum operator for the Bloch mode amplitude (6). One easily finds that $j_z(z) = m + \langle \tilde{u}_m | L_z | \tilde{u}_m \rangle(z)$ or

$$m = j_z(z) - \langle \tilde{u}_m | L_z | \tilde{u}_m \rangle(z). \quad (7)$$

This property shows that angular Bloch momentum is conserved despite angular momentum is not. The expectation value appearing in Eq.(7) is $j_u^m(z) \equiv \int_{\mathbb{R}^2} \tilde{u}_m^*(-i\partial/\partial\theta)\tilde{u}_m$ corresponding to the angular momentum associated to the θ -dependent Bloch functions \tilde{u}_m . The conservation of the angular Bloch momentum is then the result of a subtle balance between two non-conserved quantities: the conventional angular momentum $j_z(z)$ and the angular momentum $j_u^m(z)$ related to Bloch functions. The latter can be attributed to the presence of the discrete-symmetry system that acts as an angular periodic crystal. The dependence of the \tilde{u}_m Bloch functions on θ is a consequence of the angular periodicity of the system. In order to see it, it is clarifying to consider an $O(2)$ -invariant medium as the $n \rightarrow \infty$ limit of a system with discrete symmetry of n th order. In such a case, the \tilde{u}_m Bloch functions become independent of the θ angle and,

therefore, their associated angular momentum vanishes $j_u^m(z) \xrightarrow{n \rightarrow \infty} 0$. As a consequence, angular momentum becomes angular Bloch momentum $j_z(z) \xrightarrow{n \rightarrow \infty} m$ and, therefore, a constant of motion. It is then reasonable to attribute the angular momentum contribution $j_u^m(z)$ to the discrete-symmetry system acting as an angular periodic crystal. One can interpret Eq.(7) as the statement that during propagation conventional angular momentum is transferred to the discrete-symmetry system (the angular crystal), and vice-versa, in such a way angular Bloch momentum is conserved: $dj_z(z)/dz = -dj_u^m(z)/dz$.

All the previous arguments apply to every type of evolving field provided it satisfies the initial condition of having well-defined angular Bloch momentum. Conservation of angular Bloch momentum appears then as a property which occurs independently whether the solution is stationary or not. Nonlinear stationary solutions of Eq.(1) are only particular cases to which the general conservation law can be applied to. For discrete-symmetry vortices the angular-momentum-vorticity equivalence $v = j_z$ of $O(2)$ -invariant solutions is lost. Since discrete-symmetry vortices are angular Bloch modes of vorticity $v = m$, Eq.(7) establishes a new relation between vorticity and the angular momentum carried by the vortex field (as defined in Eq.(4)): $v = j_z - j_u^m$. In this way, the angular Bloch momentum concept permits to unveil the role played by angular momentum and vorticity in systems whose symmetry is no longer $O(2)$ but a point-symmetry of lesser order.

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- [1] G. A. Swartzlander and C. T. Law, Phys. Rev. Lett. **69**, 2503 (1992).
 - [2] D. N. Neshev, T. J. Alexander, E. A. Ostrovskaya, Y. S. Kivshar, H. Martin, I. Makasyuk, and Z. G. Chen, Phys. Rev. Lett. **92**, 123903 (2004).
 - [3] J. W. Fleischer, G. Bartal, O. Cohen, O. Manela, M. Segev, J. Hudock, and D. N. Christodoulides, Phys. Rev. Lett. **92**, 123904 (2004).
 - [4] A. Ferrando, M. Zacarés, P. F. de Córdoba, D. Binosi, and J. A. Monsoriu, Opt. Express **12**, 817 (2004).
 - [5] A. Ferrando, M. Zacarés, and M. A. García-March, arXiv:nlin.PS/0411005 (2004).
 - [6] N. W. Ashcroft and N. D. Mermin, *Solid state physics* (Saunders College Publishing, 1976).
 - [7] A. Ferrando, M. Zacarés, P. F. de Córdoba, D. Binosi, and A. Montero, arXiv:physics/0407029 (2004).